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Best L_p-Approximations to Continuous and Quasi-Continuous Functions by Non-Decreasing Functions on the Unit Square

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ABSTRACT. The author introduces the definition of quasi-continuity on the unit square $[0,1] \times [0,1]$. Let Q be the Banach space, under the sup-norm, of quasi-continuous functions on the unit square. Let M denote the closed convex cone in Q comprised of non-decreasing functions on the unit square. Let C be the space of continuous functions on the unit square. For $f \in Q$ and $1 < p < \infty$, let f_p denote the best L_p-approximation to f by elements of M. He shows that f_p converges uniformly as p tends to infinity to a best L_∞-approximation by elements of M. Moreover if $f \in C$, then each $f_p \in C$ and so is f_∞.

1. Introduction

We start with some introductory remarks and notations in the plane R^2. The generalization from R^2 to R^n where $n > 2$ is easy. We choose R^2 since it is much easier to visualize and understand the ideas and concepts introduced here.

Let Ω be the unit square in R^2. Let μ denote the 2-dimensional Lebesgue measure on Ω. Let σ consist of the μ-measurable subsets of Ω, and for $1 < p \leq \infty$, let $L_p = L_p(\Omega, \sigma, \mu)$. If $\bar{x} = (x_1,x_2)$ and $\bar{y} = (y_1,y_2)$ are elements of Ω, we write $\bar{x} \leq \bar{y}$ only if $x_1 \leq y_1$ and $x_2 \leq y_2$. By a function, unless we specify otherwise, we mean a real-valued function defined on Ω.

A function $g: \Omega \rightarrow R$ is said to be non-decreasing in each variable separately if $\bar{x}$, $\bar{y} \in \Omega$ and $\bar{x} = (x_1,x_2) \leq (y_1,y_2) = \bar{y}$ imply that $g(x_1,x_2) \leq g(y_1,y_2)$. Such a function is said to be non-decreasing on Ω if the following condition is also satisfied: If $\bar{x}$ is in the boundary of Ω, then

$$g(\bar{x}) = \begin{cases} \inf \ g(\bar{y}) \quad \bar{y} \leq \bar{x} \} & \bar{x} = (0,x_2) \text{ or } \bar{x} = (x_1,0) \\ \sup \{ g(\bar{y}) : \bar{y} \leq \bar{x} \} & \text{otherwise.} \end{cases} \tag{1.1}$$

Let M consist of all non-decreasing functions on Ω. Then M is closed and convex[1,p.425].

Next, we introduce the definition of the discontinuity of the first kind and the definition of quasi-continuity on Ω. This definition generalizes the definition of quasi-continuity on [0,1] as described in Darst and Sahab[2].

Definition. Let $(x_1,y_1) \in \Omega$. A function f is said to have a discontinuity of the first kind at (x_1,y_1) if given $\epsilon > 0$, there exists $\delta > 0$ and $L_1, L_2 \in R$ such that for all $(x,y) \in \Omega$ with $(x_1,y_1) \leq (x,y)$ and $d_p((x_1,y_1), (x,y)) < \delta$ we have $|f(x,y) - L_1| < \epsilon$. Also for $(x_1,y_1) \geq (x,y)$ and $d_p((x_1,y_1), (x,y)) < \delta$ we have $|f(x,y) - L_2| < \epsilon$.

We denote this by writing

$$\lim_{(x,y) \uparrow (x_1,y_1)} f(x,y) = L_1$$

and,

$$\lim_{(x,y) \downarrow (x_1,y_1)} f(x,y) = L_2 , \tag{1.3}$$

We call L_1, the lower-hand limit of f at (x_1,y_1), and L_2 the upper-hand limit of f at (x_1,y_1).

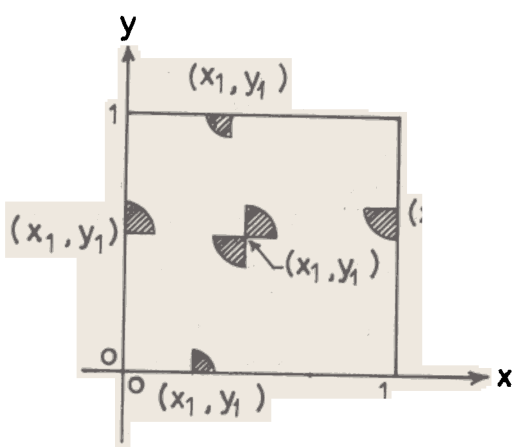

FIG. 1.1

Definition. A function f is said to be quasi-continuous on Ω if for all points $(x_1,y_1) \in \text{Int}\,\Omega$, both the lower and upper hands limits exist. For $(0,y_1), (x_1,0) \in \partial\,\Omega$, $x_1 \neq 1 \neq y_1$ the upper-hand limit must exist, and for $(1,y_1), (x_1,1) \in \partial\,\Omega$, $x_1 \neq 0 \neq$ the lower-hand limit must also exist.

This definition is consistent with the definition of a monotone non-decreasing function as we show in the next lemma.

Lemma. If $f \in M$, then $f \in Q$.

Proof. Let $(x_1,y_1) \in \text{Int}\,\Omega$. Then for $(x,y) \le (x_1,y_1)$ we have

$$L_1 = \lim_{(x,y)\uparrow(x_1,y_1)} f(x,y) = \sup\{\, f(x,y) : (x,y) \le (x_1,y_1) \,\}$$

and,

$$_2 = \lim_{(x,y)\downarrow(x_1,y_1} f(x,y) = \inf \;\; f(x,y) : (x,y) \ge (x_1,y_1) \,\}$$

Similarly, we consider points on the boundary of Ω as mentioned earlier in the definition of non-decreasing functions.

As done by Darst and Sahab[2] we consider every f in Q as bounded Lebesgue measurable function, we we let

$$[f] = \{\, g{:}g \text{ is measurable}, f = g \text{ a.e.} \} \tag{1.4}$$

be the corresponding elements of L_∞.

A function $f \in Q$ is zero <=> for every $(x_1,y_1) \in \text{Int}\,\Omega$,

$$\lim_{(x,y)\uparrow(x_1,y_1)} f(x,y) = \lim_{(x,y)\downarrow(x_1,y_1)} f(x,y) = 0.$$

Next, let Q^* denote the space of functions $f \in Q$ such that

$$f(0,y_1) = \lim_{(0,y)\uparrow(0,y_1)} f(0,y)$$

and

$$f(x_1,0) = \lim_{(x,0)\downarrow(x_10)} f(x,0)$$

where $0 \le x_1,y_1 < 1$, and

$$f(x_1,y_1) = \lim_{(x,y)\uparrow(x_1,y_1)} f(x,y)$$

For all $(x,y) \in \text{Int}\,\Omega \cup \{(1,y_1), (x_1,1) : 0 < x_1,y_1 \le 1\}$.

Clearly we have a linear isometry between Q^* and the embedding of Q in $L_\infty(\Omega)$.

Now, let P denote the set of square partitions π partitioning Ω into n squares of eaual areas as shown in Fig. 1.2.

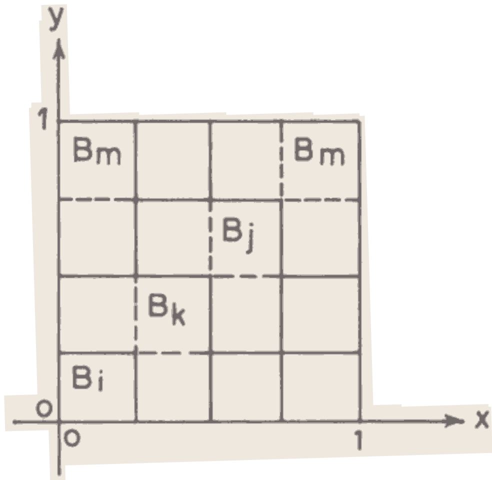

FIG. 1.2

Let I_B denote the indicator function of a square $B \leq \Omega$, i.e., $I_B(x,y) = 1$ if $(x,y) \epsilon B$ and $I_B(x,y) = 0$ otherwise.

Denote by S^* the dense linear subspace of Q comprised of all steps functions of

$$= \sum_{i=1}^{n} a_i I_{B_i},\ a_i \epsilon R,\ \Omega = \bigcup_{i=1}^{n} B_i \text{ with } B_i \cap B_j = \emptyset, i \neq, j.$$

It was shown by Darst and Sahab[2] that for n [0,1], f_p converges uniformly as $p \rightarrow \infty$ to a best L_∞ – approximation to f by monotone non-decreasing functions on [0,1].

From now on, we consider Q^*, and we look at best L_p-approximations to elements of Q^* by elements of $M^* = M \cap Q^*$.

2. Basic Generalizations

In this section we obtain some results for approximations on Ω. These results are established by modifying the proofs of the corresponding results in Drast and Sahab[2] for functions on [0,1].

It is very important at this stage to be familiar with the concepts, results and proofs in Drast and Sahab[2], in order to understand the briefings mentioned in what follows of this section.

Let $\pi = \bigcup_{i=1}^{n} B_i$ with $B_i \cap B_j = \emptyset$ be a partition of Ω into a set of disjoint sub-squares of equal area.

Let $X = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ be a finite partially ordered set in the plane. The literature in Darst and Sahab[2,pp.10-11] which is extracted from Ubhaya[6] carries over in the same manner.

Lemma 1. If $f \in S_\pi^*$, then $f_p \in S_\pi^*$ for all p, $1 < p < \infty$, where $S_\pi^* = S_\pi \cap Q^*$

Proof. Suppose f_p is not constant a.e. on some subsquare B_j. Then let

$$\ell = \text{essinf}\,\{f_p(\bar{t}) : \bar{t} \in B_j\}\ ,$$

and

$$u = \text{essup}\,\{f_p(\bar{t}) : \bar{t} \in B_j\}\ .$$

Clearly $\ell < u$. Choose $\zeta \in [\ell, u]$ such that

$$|f_i - \zeta| = \inf\{\,|f_j - r| : r \in [\ell, u]\}\ .$$

Then the monotone non-decreasing function defined by

$$\begin{aligned} f_p^*(\bar{t}) &= \zeta, & \bar{t} \in B_j\ , \\ &= f_p(\bar{t}) & \text{otherwise,} \end{aligned}$$

is a better best L_p – approximation to f since

$$\|f - f_p^*\| = \left| \sum_{i=1}^{n} \int_{B_i} |f_i - f_p(\bar{t})|^p\, d\bar{t} + \int_{B_j} |f_j - \zeta|^p\, d\bar{t} \right|^{\frac{1}{p}}$$

$$< \left| \sum_{\substack{i=1 \\ i \neq j}}^{n} \int_{B_i} |f_i - f_p(\bar{t})|^p\, d\bar{t} + \int_{B_j} |f_j - f_p(\bar{t})|^p\, d\bar{t} \right|^{\frac{1}{p}}$$

or,

$$\|f - f_p^*\|_p < \|f - f_p\|_p\ .$$

Contradiction! Hence f_p must be constant everywhere on B_j and $f_p \in S_\pi^*$

Theorem 1. Let $f \in S^*$ be given by $f = \sum_{i=1}^{n} f_i I_{B_i}$

For every p, $1 < p < \infty$, let $w_p = \{w_{p,i}\}_{i=1}^{n}$ be defined by

$$w_{p,i} = A(B_i) = \text{Area of } B_i \qquad (2.3)$$

for all i. Let $g_p = \{g_{p,i}\}_{i=1}^{n}$ be as defined by Darst and Sahab[2] and Shilov and Gurevich[4].

$$g_{p,i} = \max_{\{U:i\in U\}} \min_{\{L:i\in L\}} u_p\,(L \cap U)$$

$$\min_{\{L:i\in L\}} \max_{\{U:i\in U\}} u_p\,(L \cap U)$$

Then f_p is given by

$$f_p = \sum_{i=1} g_{p,i} I_{B_i}$$

Proof. By the last lemma, we have $f_p \in S^*_\pi$. For every i, let

$$\bar{t}_i = (x_i,y_i) = \text{Center of } B_i,$$

and let $X = \{\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n\}$. Then X is partially ordered. Consider $\{f_i\}^n_{i=1}$ as a finite real valued function defined on X, and let $h = \{h_i\}^n_{i=1}$ be a monotone non-decreasing function on X. The rest of the proof follows from Theorem 2 of Darst and Sahab[2], through simple modifications as was done in Lemma 1.

Theorem 2. Let $f \in S^*_\pi$ and let f_p be as given in Theorem 1. Then f_p converges as $p \to \infty$ to the monotone non-decreasing function $f_\infty \in S^*_\pi$ given by

$$f_\infty = \sum_{i=1}^{n} g_{\infty,i} I_{B_i} \tag{2.6}$$

where $g_{\infty,i} = \lim_{p\to 1} g_{p,i} = \max_{\{U:i\in U\}} \min_{\{L:i\in L\}} \mu_\infty(L \cap U)$.

Proof. Follow the proof of Theorem 3 in Darst and Sahab[2] with the right modification.

Next, we state some remarks, definitions and results which are generalizations of their counterparts discussed in Darst and Sahab[2].

Remark 1. If $f \in S^*_\pi$. We denote it by f_π. Similarly, we let

$$f_{\pi,p} = (f_\pi)_p ,$$

and,

$$f_{\pi,\infty} = (f_\pi)_\infty = \lim_{p\to\infty} f_{\pi,p} .$$

Remark 2. (a) Let f and g be elements of Q^* such that $f \le g$. Then

$$f_p \le g_p$$

for all p, $1 < p < \infty$.

(b) For every constant c,

$$(f + c)_p = f_p + c .$$

Definition. Let $f \in Q^*$ and let $\pi = \{B_i\}^n_{i=1}$ be a partition of Ω. The oscillation of f over B_i is defined by

$$\tilde{\sigma}[f,B_i] = \sup\{f(\bar{x}) - f(\bar{y}) \ : \ \bar{x},\bar{y} \in B_i \tag{2.7}$$

and the oscillation of f over π is defined by

$$\tilde{\sigma}(f,\pi) = \max_{1\le i\le n} \{\tilde{\sigma}[f,B_i]\} . \tag{2.8}$$

Lemma 2. Let $\pi' = \{B_i'\}_{i=1}^{n'}$ be a refinement of $\pi = \{B_i\}_{i=1}^{n}$; $n < n'$ (written $\pi < \pi'$). Then $\tilde{\sigma}(f,\pi') \leq \tilde{\sigma}(f,\pi)$.

Remark 3. Let $f \in Q^*$ and let $\epsilon > 0$ be given. Then there exists a partition π such that $\tilde{\sigma}(f,\pi) < \epsilon$,

Moreover, if $0 < \epsilon' < \epsilon$, then there is a refinement π' of π such that $\tilde{\sigma}(f,\pi') < \epsilon'$. Hence $\tilde{\sigma}(f,\pi)$ can be made as small as possible by refining π. We denote this by writing

$$\lim_{\pi} \tilde{\sigma}(f,\pi) = 0 . \tag{2.9}$$

Definition. Let $f \in Q^*$ and $\pi = \{B_i\}_{i=1}^{n}$ be a partition of Ω. For every i, $1 \leq i \leq n$, let

$$t_i = \inf_{(\bar{x},\bar{y}) \in B_i} f(\bar{x},\bar{y}) , \tag{2.10}$$

and

$$\tau_i = \sup_{(\bar{x},\bar{y}) \in B_i} f(\bar{x},\bar{y}) \tag{2.11}$$

Then the two expressions

$$\underline{f}_\pi = \sum_{i=1}^{n} t_i I_{B_i} ,$$

and,

$$\bar{f}_\pi = \sum_{i=1}^{n} \tau_i I_{B_i} ,$$

are called the lower and upper step functions generated by π, respectively.

Definition. For $\underline{f}_\pi$ and $\bar{f}_\pi$ defined above, let

$$\bar{f}_{\pi,p} = (\bar{f}_\pi)_p ;$$

$$\underline{f}_{\pi,p} = (\underline{f}_\pi)_p ; \tag{2.15}$$

and,

$$\bar{f}_{\pi,\infty} = (\bar{f}_\pi)_\infty = \lim_{p \to \infty} \bar{f}_{\pi,p} , \tag{2.16}$$

$$\underline{f}_{\pi,\infty} = (\underline{f}_\pi)_\infty = \lim_{p \to \infty} \underline{f}_{\pi,p} . \tag{2.17}$$

The proofs of the following two lemmas can be generalized easily from the proofs of Lemma 3 and Lemma 4 in Darst and Sahab[2], respectively.

Lemma 3. For all p, $1 < p < \infty$, we have

$$0 \leq \bar{f}_{\pi,p} - \underline{f}_{\pi,p} \leq \tilde{\sigma}(f,\pi) ,$$

and,

$$0 \leq \overline{f}_{\pi,\infty} - \underline{f}_{\pi,\infty} \leq \widetilde{\sigma}(f,\pi) .$$

Lemma 4. Let $f \in Q^*$ and let $\pi < \pi'$. Then

$$\underline{f}_{\pi,p} \leq \underline{f}_{\pi',p} \leq \overline{f}_{\pi',p} \leq \overline{f}_{\pi,p} \leq \underline{f}_{\pi,p} + \widetilde{\sigma}(f,\pi) ,$$

$$\underline{f}_{\pi,\infty} \leq \underline{f}_{\pi',\infty} \leq \overline{f}_{\pi',\infty} \leq \overline{f}_{\pi,\infty} \leq \underline{f}_{\pi,\infty} + \widetilde{\sigma}(f,\pi) .$$

Finally, in this section, we state the following Theorem.

Theorem 3. Let $f \in Q^*$ with best monotone L_p – approximation f_p. Then

$$\lim_{\pi} \overline{f}_{\pi,p} = \lim_{\pi} \underline{f}_{\pi,p} = f_p ,$$

$$\lim_{\pi} \overline{f}_{\pi,\infty} = \lim_{\pi} \underline{f}_{\pi,\infty} = f_\infty = \lim_{p\to\infty} f_p$$

Proof. See the proofs of Theorems 4 and 5 in [2, pp.18-19].

3. The Case When f Is Continuous

We choose to write the full proof of the following theorem because of the nature of the work involved here.

Theorem 4. Let f be continuous on Ω, then f_p is continuous, $1 < p < \infty$.

Proof. Let (x,y) be an interior point of Ω and let it be fixed. Let $\epsilon > 0$ be given. Then

$$|f_p(x,y) - f_p(x',y')| \leq |f_p(x,y) - \overline{f}_{\pi,p}(x,y| + |\overline{f}_{\pi,p}(x,y) - \overline{f}_{\pi,p}(x',y')| + |\overline{f}_{\pi,p}(x',y') - f_p(x',y')| \quad (3.1)$$

Since Theorem 3.1 implies that

$$f_p(x,y) = \lim_{\pi} f_{\pi,p}(x,y)$$

for all $(x,y) \in \Omega$, we can choose a partition $\pi = \{B_i\}_{i=1}^n$ with the following

(1) Each of the first and third term on the right hand side of (3.1) is less than $\epsilon/3$.

(2) Suppose $\overline{f}_\pi$ is given by

$$\overline{f}_\pi = \sum_{i=1}^{n} \tau_i I_{B_i} . \quad (3.2)$$

Then by the uniform continuity of f over Ω we can have

$$|\tau_i - \tau_{i-1}| < \epsilon / 9 \quad (3.3)$$

for all i = 2,3 ,..., n

Thus, (3.1) becomes

$$|f_p(x,y) - f_p(x',y')| < \epsilon/3 + |\bar{f}_{\pi,p}(x,y) - \bar{f}_{\pi,p}(x',y')| + \epsilon/3 \qquad (3.4)$$

for all $(x',y') \in \Omega$. We need to show that there exists a real number $\delta > 0$ such that

$$|\bar{f}_{\pi,p}(x,y) - \bar{f}_{\pi,p}(x',y')| < \epsilon/3 \qquad (3.5$$

for all $(x',y') \in N_\delta(x,y)$, where $N_\delta(x,y)$ is an open disk of radius δ centered at (x,y).

We start by observing first that if f is given by (3.2), then $\bar{f}_{\pi,p}$ must be given by

$$\bar{f}_{\pi,p} = \sum_{i=1}^{n} \gamma_i I_{B_i}. \qquad (3.6$$

We now have two cases to consider:

Case 1. $(x,y) \in \text{Int}(B_j)$ for some $j \leq n$. Then it follows that

$$|\bar{f}_{\pi,p}(x,y) - \bar{f}_{\pi,p}(x',y')| = |\gamma_j - \gamma_j| = 0$$

for all $(x',y') \in B_j$. Let $\delta = \min(\delta_1, \delta_2)$ [see Fig. (3.1)]

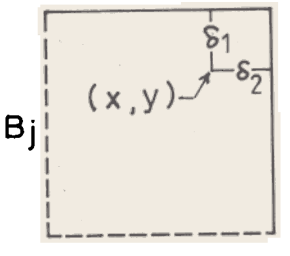

FIG. 3.1

Then (3.4) becomes

$$|f_p(x,y) - f_p(x',y')| < 2\epsilon/3$$

for all $(x',y') \in N_\delta(x,y)$ which implies the continuity of f_p at x in this case.

Case 2. (x,y) lies on the boundary of B_j but it is none of the vertices of B_j for some $j \leq n$.

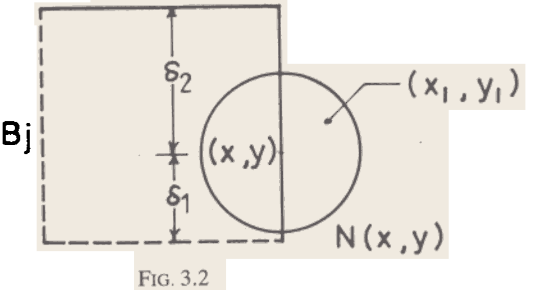

FIG. 3.2

Then it follows from (3.6) that

$$|\overline{f}_{\pi,p}(x,y) - \overline{f}_{\pi,p}(x',y')| = |\gamma_j - \gamma_j| = 0$$

for all $(x',y') \in B_j \cap N_\delta(x,y)$ where $\delta = \min(\delta_1,\delta_2)$ and δ_1 and δ_2 as shown in Fig. (3.2).

Now, consider $(x',y') \in B_j^c \cap N_\delta(x,y)$, and suppose that

$$|f_{\pi,p}(x',y') - \overline{f}_{\pi,p}(x,y)| = \overline{f}_{\pi,p}(x',y') - \overline{f}_{\pi,p}(x,y) = \gamma_{j+1} - \gamma_j > \epsilon / 3 .$$

Then, we obtain

$$\epsilon / 3 < \gamma_{j+1} - \gamma_j = (\gamma_{j+1} - \tau_{j+1}) + (\tau_{j+1} - \tau_j) + (\tau_j - \gamma_j)$$

Since $\tau_{j+1} - \tau_j < \epsilon / 9$ by (3.3), we may assume without loss of generality that

$$\gamma_{j+1} - \tau_{i+1} > \epsilon / 9 .$$

In such a case let

$$\gamma^*_{j+1} = \gamma_{j+1} - \epsilon / 9$$

Hence,

$$\gamma^*_{j+1} - \gamma_j = (\gamma_{j+1} - \gamma_j) - \epsilon / 9 > \epsilon / 3 - \epsilon / 9 = 2\epsilon / 9 > 0 .$$

Now, let $\overline{f}^*_{\pi,p}$ be the non-decreasing step function defined on Ω by

$$\overline{f}^*_{\pi,p} = \sum_{\substack{i=1 \\ i \neq j+1}}^{n} \gamma_i I_{B_i} + \gamma^*_{j+1} I_{B_{j+1}}$$

Then,

$$\|\overline{f}^*_{\pi,p} - \overline{f}_\pi\|_p^p = \sum_{\substack{i=1 \\ i \neq j+1}}^{n} |\gamma_i - \tau_i|^p A(B_i)$$

$$+ | \gamma^*_{j+1} - \tau_{j+1}|^p A(B_{j+1}) ,$$

while,

$$\|\overline{f}_{\pi,p} - \overline{f}_\pi\|_p^p = \sum_{i=1}^{n} |\gamma_i - \tau_i|^p A(B_i)$$

But notice that (3.7) and (3.8) imply that

$$\gamma^*_{j+1} - \tau_{j+1} = \gamma_{j+1} - \epsilon / 9 - \tau_{j+1}$$

$$= (\gamma_{j+1} - \tau_{j+1}) - \epsilon / 9 > \epsilon / 9 - \epsilon 9 = 0 ,$$

or,

$$0 < \gamma^*_{j+1} - \tau_{j+1} < \gamma_{j+1} - \tau_{j+1}$$

or.

$$|\gamma^*_{j+1} - \tau_{j+1}|^p < |\gamma_{j+1} - \tau_{j+1}|^p ,$$

which implies upon comparing (3.10) and (3.11) that

$$\|\bar{f}^*_{\pi,p} - \bar{f}_\pi\|_p < \|\bar{f}_{\pi,p} - \bar{f}_\pi\|_p .$$

Contradiction! Thus, our assumption is not correct and hence we must have

$$|f_{\pi,p}(x,y) - f_{\pi,p}(x',y')| < \epsilon / 3 ,$$

for all $(x',y') \in B_j^c \cap N_\delta(x,y)$. Therefore (3.4) becomes

$$|f_p(x,y) - f_p(x',y')| < 2\epsilon / 3 + \epsilon / 3 = \epsilon ,$$

for all $(x',y') \in N_\delta(x,y)$.

Case 3. (x,y) is the vertex of a square (Fig. 3.3)

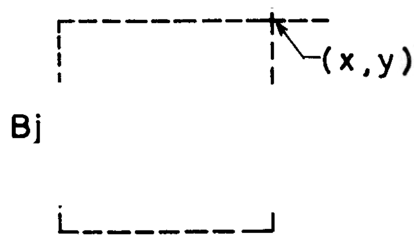

FIG. 3.3

This case can be treated by splitting $N_\delta(x,y)$ to four different parts, and then applying the steps of case 2. This completes the proof

Corollary 1. The function $f_\infty = \lim_{p\to\infty} f_p$ is continuous on Ω when f is continuous on Ω.

Proof. Since f_∞ is the uniform limit of continuous functions, it must be continuous.

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أفضـــل تقــريبـات L_p للـــدوال المتصلة وشبـــه المتصلة باستخـــدام الـدوال غـــير التـــناقصية عـــلى الوحــــدة المربعـــة

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في هذا البحث نعرف شبه الاتصال على الوحدة المربعة ، ونثبت أن تقريبات L_p تتقارب بانتظام إلى إحدى تقريبات L_∞ . علاوة على ذلك نثبت أن أفضل تقريبات L_p لأي دالة متصلة هي أيضاً دالة متصلة وكذلك تقريب L_∞ الناتج من تقاربات L_p .